Simulation methods in ruin models with non-linear dividend barriers

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In the framework of classical risk theory we consider the process $R_t = u + c\,t - \sum_{i=1}^{N(t)} X_i$, where c is a constant premium density, $N(t)$ denotes a homogeneous Poisson process with intensity λ which counts the claims up to time t, and the claim amounts X_i are iid random variables with distribution function $F(y)$. In this context R_t represents the surplus of an insurance portfolio at time t and $u = R_0$ denotes the initial capital (see e.g. Asmussen [3]). Furthermore we assume $\mu = E(X_i) < \infty$ and $c > \lambda \int_0^\infty y\,dF(y)$. A reasonable modification of this model is the introduction of a (time-dependent) dividend barrier b_t, i.e. whenever the surplus R_t reaches b_t, dividends are paid out to the shareholders with intensity $c - db_t$ and the surplus remains on the barrier, until the next claim occurs. Thus the risk process develops according to

$$\begin{aligned} dR_t &= c\,dt - dS_t \quad &\text{if} \quad R_t < b_t \\ dR_t &= db_t - dS_t \quad &\text{if} \quad R_t = b_t, \end{aligned}$$

where we have used the abbreviation $S_t = \sum_{i=1}^{N(t)} X_i$. Together with the initial capital $R_0 = u$, $0 \le u < b_0 < \infty$, this determines the risk process $(R_t)_{t\ge 0}$. Quantities of particular interest in this context are the probability of survival $\phi(u,b) = Pr\,(R_t \ge 0\ \forall t > 0\,|R_0 = u, b_0 = b)$ and the expected sum of discounted dividend payments $W(u,b)$.

Dividend barrier models have a long history in risk theory (see e.g. [4]).Whereas $\phi(u,b) = 0$ for all $0 \le u \le b$ in the case of a horizontal dividend barrier, monotonically increasing dividend barriers provide a much richer structure. The case of linear dividend barriers is fairly well understood: Gerber [5] derived an upper bound for the probability of ruin for $b_t = b + at$ (where a and b are constants) by martingale methods and in [6] he obtained exact solutions for the probability of ruin and the expected sum of discounted dividend payments $W(u,b)$ for exponentially distributed claim amounts; this result was generalized by Siegl and Tichy [7] to arbitrary Erlang claim amount distributions, see also Albrecher and Tichy [2].

In [1] non-linear dividend barrier models of the type

$$b_t = \left(b^m + \frac{t}{\alpha}\right)^{1/m} \qquad (\alpha, b > 0,\ m > 1)$$

were introduced and integral equations for $\phi(u,b)$ and $W(u,b)$ were derived. The existence and uniqueness of the corresponding solutions was discussed and techniques for numerical solutions were developed and tested for the case of an exponential claim size distribution.

In this paper we extend the results of [1] in various directions in that we allow for more general claim size distributions and continuously compounded interest on the free reserve. Both the probability of survival and the expected sum of discounted dividend payments can be identified as the fixed point of a contracting integral operator A of the form

$$Ag(u,b) = \int_0^\infty f_1(u,b,t) \int_0^{f_2(u,b,t)} g\Big(f_3(u,b,t,z), f_4(b,t)\Big) dF(z)dt + f_5(u,b) \qquad (1)$$

in a suitable Banach space of functions, where $f_1, \ldots, f_5$ are computable functions of the corresponding variables. This fact is used to formulate number-theoretic solution methods:

An approximation for the solution can for instance be obtained by recursively applying the integral operator (1) k times to a suitably chosen starting function. This leads to a $2k$-dimensional integral for the corresponding function for given values of u and b, which can then be calculated numerically by Monte Carlo and Quasi-Monte Carlo methods.

Another solution technique used in this paper is to discretize the domain of u and b and after assigning a suitable initial value to each discretization point (u_i, b_j), the operator (1) is applied sequentially to each point (u_i, b_j). The resulting two-dimensional integral for each point is then calculated by Monte Carlo and Quasi-Monte Carlo methods. This procedure is then repeated several times (depending on the desired accuracy of the approximative solution). The numerical integration by MC and QMC requires the evaluation of the function g for values of the arguments that may lie outside the discretization grid; those are obtained by interpolation.

The risk process can also be simulated directly by sampling N paths (starting at $R_0 = u$) of the process and take the arithmetic mean of the outcomes of each path as an approximation to the desired solution. For each path the times between successive claims are exponentially distributed and the claim sizes are distributed according to $F(z)$. The random numbers needed for the simulation procedure are generated by pseudorandom and quasi-random sequences, respectively.

These methods are also applied for a model with a constant absorbing upper barrier, where the surplus process is stopped if it has reached a given value and accordingly no dividend payments can occur after this event. Economically this can be interpreted that the company will then decide to pursue other forms of investment strategies. Mathematically, this modification of the model has some attractive features (e.g. the process stops in finite time with probability 1).

The main focus of our paper is on the quantitative and qualitative comparison of the performance of the various numerical solution techniques for each of the models described above. In particular, the efficiency gain obtained by implementing low-discrepancy sequences (Halton, Sobol, Faure and Niederreiter sequences) instead of pseudorandom sequences is investigated.

References

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