Quasi-Monte Carlo Algorithms with Applications in Numerical Analysis and Finance

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Quasi-Monte Carlo methods

Integrals approximated by a discrete sum over N (quasi-)random points:

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MC methods: x_i random points

QMC methods: x_i low-discrepancy sequences

Low discrepancy sequences: deterministic point sequences $\{x_i\}_{1\leq n\leq N} \in [0,1)^s$, good uniform distribution

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<u>discrepancy of the point set S</u>

$$D_N(S) = \sup_{\mathcal{J}\subseteq[0,1]^s} \left| \frac{A(\mathcal{J},S)}{N} - \lambda_s(\mathcal{J}) \right|$$

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Koksma-Hlawka inequality (f of bounded Variation):

$$\left| \frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_{[0,1)^s} f(u)du \right| \leq V([0,1)^s, f) D_N^* (x_1, \ldots, x_N) \ .$$

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- (t, s) nets in base b (Niederreiter, Sobol, Faure): net-like structure $\rightarrow$ best possible uniform distribution on elementary intervals

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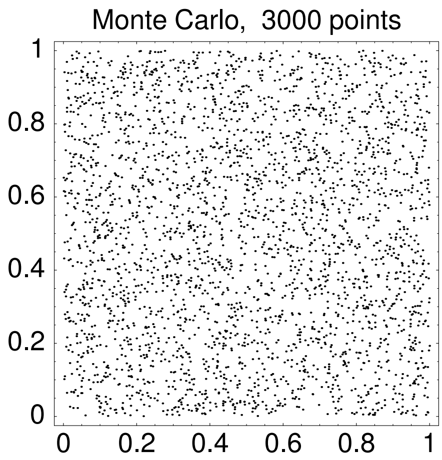

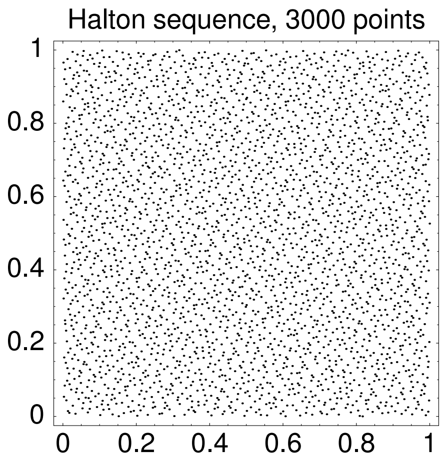

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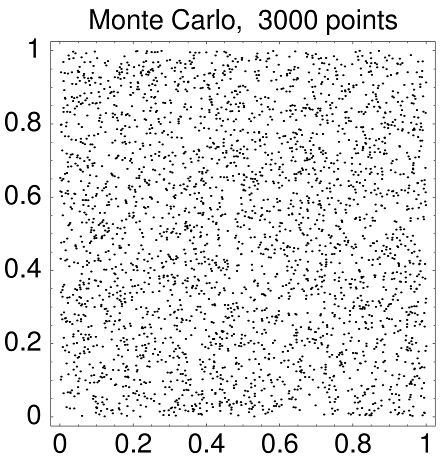
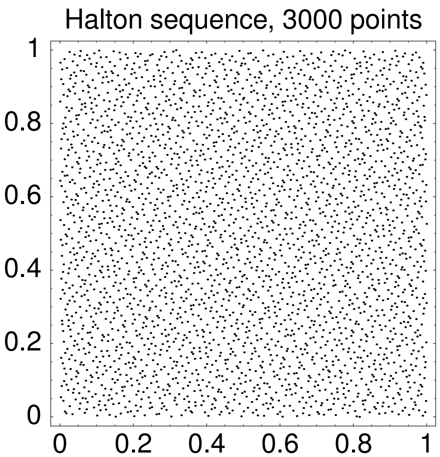
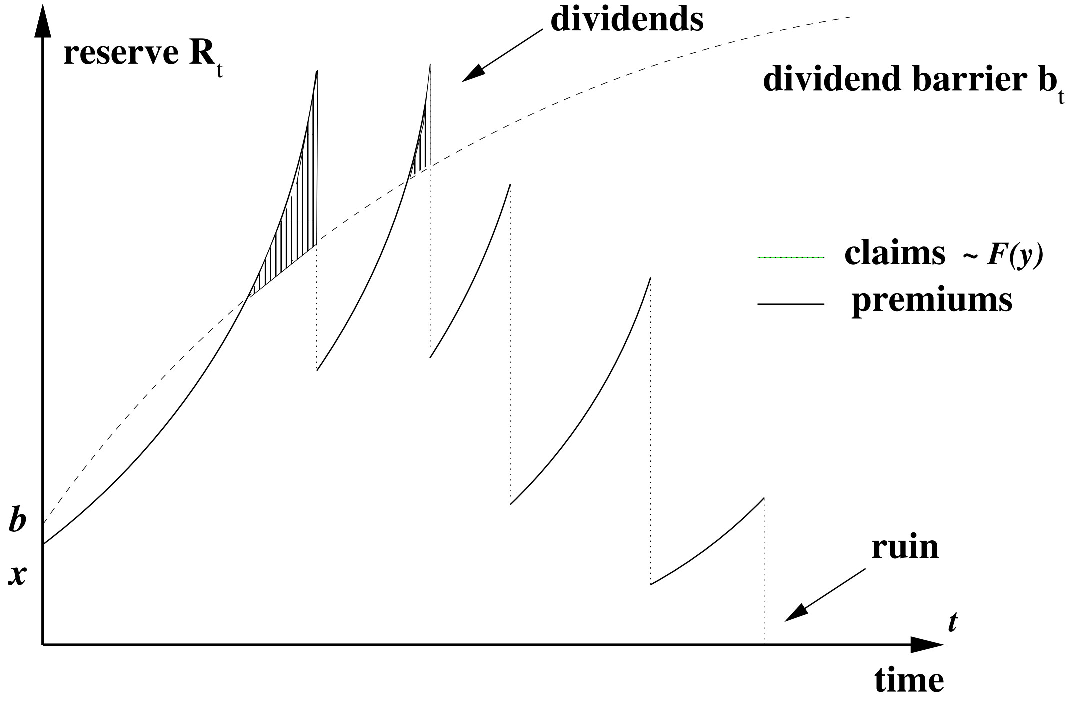




Problem: correlations between elements

Risk Model with constant interest force and non-linear dividend barrier



$\phi(u, b) \ldots$ probability of survival

$W(u, b) \ldots$ expected value of discounted dividend payments

Integro-differential equation for $W(u,b)$:

$$(c+i\,u)\,\frac{\partial W}{\partial u}+\frac{1}{\alpha\,m\,b^{m-1}}\,\frac{\partial W}{\partial b}-(i+\lambda)\,W+\lambda\int_0^u W(u-z,b)\,dF(z)=0$$

with boundary condition $\frac{\partial W}{\partial u}\big|_{u=b}=1$.

Solution is fixed point of integral operator $\Rightarrow$ apply it iteratively or recursively

$$\begin{aligned}Ag(u,b)&=\int_0^{t^*}\lambda e^{-(\lambda+i)t}\int_0^{(c'+u)e^{it}-c'} g\left((c'+u)e^{it}-c'-z,\left(b^m+\frac{t}{\alpha}\right)^{1/m}\right)dF(z)dt\\&+\int_{t^*}^{\infty}\lambda e^{-(\lambda+i)t}\int_0^{\left(b^m+\frac{t}{\alpha}\right)^{1/m}} g\left(\left(b^m+\frac{t}{\alpha}\right)^{1/m}-z,\left(b^m+\frac{t}{\alpha}\right)^{1/m}\right)dF(z)dt\\&+\int_{t^*}^{\infty}\lambda e^{-\lambda t}\int_{t^*}^{t} e^{-is}\left((c+i\,u)e^{is}-\frac{1}{m\alpha\left(b^m+\frac{s}{\alpha}\right)^{1-1/m}}\right)ds\,dt,\end{aligned}$$

$\Longrightarrow$ Solution is just a high-dimensional integration problem.

Numerical solution of delayed differential equations using QMC methods

1. Sketch of the numerical solution
2. The RKQMC solution methods (Hermite Interpolation, QMC methods)
3. Convergence proofs
4. Numerical examples

The problem

Heavily varying delay differential equations (DDE) or DDE with heavily varying solutions.

$$
\begin{aligned}
y'(t) &= f\left(t, y(t), y(t-\tau_1(t)), \ldots, y(t-\tau_k(t))\right), && \text{for } t \geq t_0,\ k \geq 1, \\
y(t) &= \phi(t), && \text{for } t \leq t_0\ ,
\end{aligned}
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with

$f(t, y(t), y_{ret}(t)) \ldots$piecewise smooth in y and y_{ret}, bounded and Borel measurable in t

$y(t) \ldots$solution, d-dimensional real-valued function

$\tau_1(t), ..., \tau_k(t) \ldots$cont. delay functions, bounded from below by $\tau_0 > 0$, satisfy $t_1 - \tau_j(t_1) \leq t_2 - \tau_j(t_2)$ for $t_1 \leq t_2$

$\phi(t) \ldots$initial function, cont. on $\left[\inf_{t_0 \leq t, 1 \leq j \leq k} (t - \tau_j(t))\, , t_0\right]$.

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  Idea (Stengle, Lécot): integrate over whole step size
  - Runge Kutta: Integration over y and t discretized
  - RK(Q)MC: Integration over y discretized, numerical Integration in t (using MC or QMC integration to minimize the error)

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$$z_j(t) := y(t - \tau_j(t)) = \begin{cases} \phi(t - \tau_j(t)), & \text{if } t - \tau_j(t) \leq t_0 \\ P_q(t - \tau_j(t); (y_i); (y_i')) & \text{otherwise} \end{cases}$$

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DDE transforms to a ODE:

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Solution has to be piecewise $r/2$-times continuously differentiable in t. Resulting ODE has to fulfill requirements for RKQMC methods (Borel-measurable in t, continously differentiable in $y(t)$).

Runge Kutta QMC methods for ODE

G. Stengle, Ch. Lécot, I. Coulibaly, A. Koudiraty

$$y'(t) = f(t, y(t)), \qquad 0 < t < T, \qquad y(0) = y_0$$

f smooth in y, bounded and Borel measurable in t.

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$$y_{n+1} = y_n + \frac{h_n}{s!N} \sum_{0 \le j < N} G_s\left(\bar{t}_{j,n}; y\right)$$

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$$G_1\left(u; y\right) = f\left(u, y\right)$$

$$G_2\left(\bar{u}; y\right) = f\left(\bar{u}_1, y\right) + \frac{1}{\beta} f\left(\bar{u}_2, y\right)) + \frac{1}{\alpha} f\left(\bar{u}_2, y + \alpha h_n f\left(\bar{u}_1, y\right)\right))$$

$$G_3\left(\bar{u}; y\right) = a_1 f(\bar{u}_1, y) + \sum_{l=1}^{L_2} a_{2,l} f\left(\bar{u}_2, y + b_{2,l} h_n f(\bar{u}_1, y)\right) + $$

$$+ \sum_{l=1}^{L_3} a_{3,l}\left(\bar{u}_3, y + b_{3,l}^{(1)} h_n f\left(\bar{u}_1, y\right) + b_{3,l}^{(2)} h_n f\left(\bar{u}_2, y_n + c_{3,l} h_n(\bar{u}_1, y_n)\right)\right)$$

RKQMC for Volterra functional equations

$$
\begin{aligned}
y'(t) &= f(t, y(t), z(t)) & t &\geq t_0 \\
z(t) &= (Fy)(t) & & \\
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RKQMC method ($n \geq 0$):

$$\begin{aligned} y_{n+1} &= y_n + h_n \sum_{i=1}^{N} G_s\left(t_{n,i}; y_n, \tilde{z}(t)\right) \\ \tilde{z}(t) &= (\tilde{F} y_j)(t) \end{aligned}$$

Convergence: RKQMC for DDE

Theorem[K., 2002] If

(i) RKQMC method converges for ordinary differential equations with order p

(ii) the increment function G_s of the method and F are Lipschitz

(iii) the interpolation fulfills a Lipschitz condition

(iv) Hermite interpolation is used with order r

(v) the initial error $||e_0||$ vanishes

then the method converges.

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then the method converges. If at least ii and iii hold, the error is bounded by

$$\|e_{j+1}\| \leq \|e_0\| \, e^{\mathcal{L}t_j} + \frac{(e^{t_j\mathcal{L}} - 1)}{\mathcal{L}} \left(\left\| E_G^{\mathrm{ODE}} \right\| + \mathcal{L}_1 \mathcal{L}_2 \left\| E_r^{\mathrm{interpol}} \right\| \right) \ .$$

Special case: one retarded argument

Choose the RKQMC method:

- G_s Lipschitz ($\mathcal{L}_2$) in 2^{nd} and 3^{rd} argument, bounded variation (in the sense of Hardy and Krause).
- $\exists\ c_1,\ c_2,\ c_3$ such that for a $p > 0$

$$\begin{aligned} \text{loc. trunc. error} \ \ \|\epsilon_n\| &\leq c_1(h_n)h_n^p \\ \text{RK error} \ \ \|\delta_n\| &\leq c_2(h_n)\,\|e_n\| \\ \text{QMC error} \ \ \|d_n\| &\leq c_3(h_n)D_N^*(S) \end{aligned}$$

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Then the error $\|e_n\| = \|y_n - y(t_n)\|$ of the method is bounded from above by:

$$\|e_n\| \leq \|e_0\|\, e^{t_n\left(c_2 + \frac{\mathcal{L}_2}{s!}\right)} + \frac{e^{t_n\left(c_2 + \frac{\mathcal{L}_2}{s!}\right)} - 1}{c_2 + \frac{\mathcal{L}_2}{s!}} \cdot \\ \cdot \left\{ c_3 D_N^*(X) + \frac{\mathcal{L}_2}{s!} M H^q + c_1 H^p \right\}$$

Numerical examples

$$
\begin{aligned}
y'(t) &= 3y(t-1)\sin(\lambda t) + 2y(t-1.5)\cos(\lambda t), \quad t \geq 0 \\
y(t) &= 1, \quad t \leq 0 \,,
\end{aligned}
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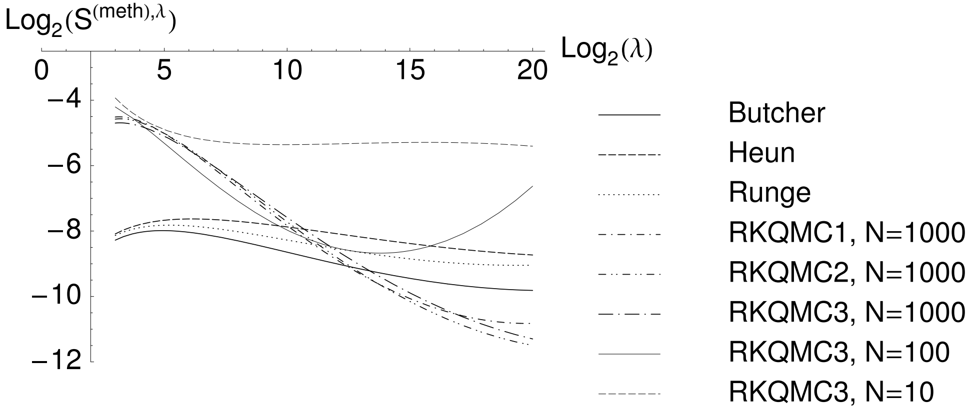

Fig: Error of the RK and RKQMC methods, $h_n = 0.001$ for RK, $h_n = 0.01$ for RKQMC

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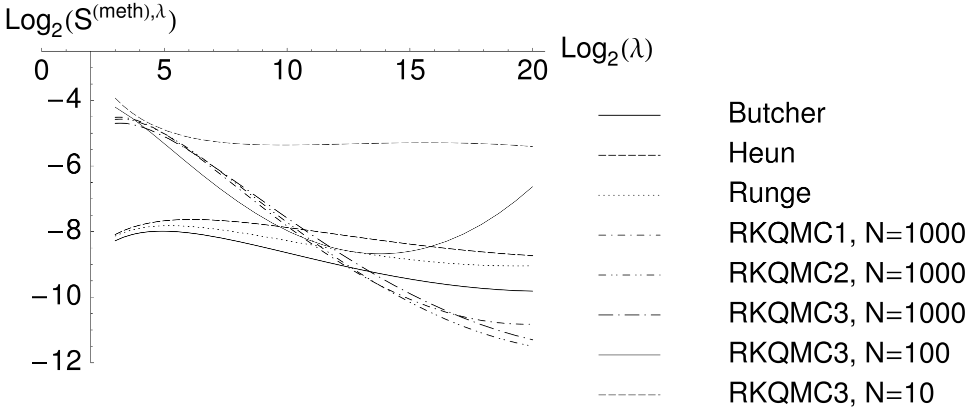

Fig: Error of the RK and RKQMC methods, $h_n = 0.001$ for RK, $h_n = 0.01$ for RKQMC

slowly varying (small λ): conventional RK better
rapidly varying (high λ): RKQMC outperform higher order RK

Advantage of RKQMC for unstable DDE

$$y'(t) = \pi\frac{\lambda}{2}\left(y\left(t-2-\frac{3}{2\lambda}\right)-y\left(t-2-\frac{1}{2\lambda}\right)\right), \quad t \geq 0$$
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Exact Solution: $y(t) = sin(\lambda t \pi)$

- No discontinuities in any derivative
- k-th derivative only bounded by $\lambda^k \rightarrow$ "exploding" error

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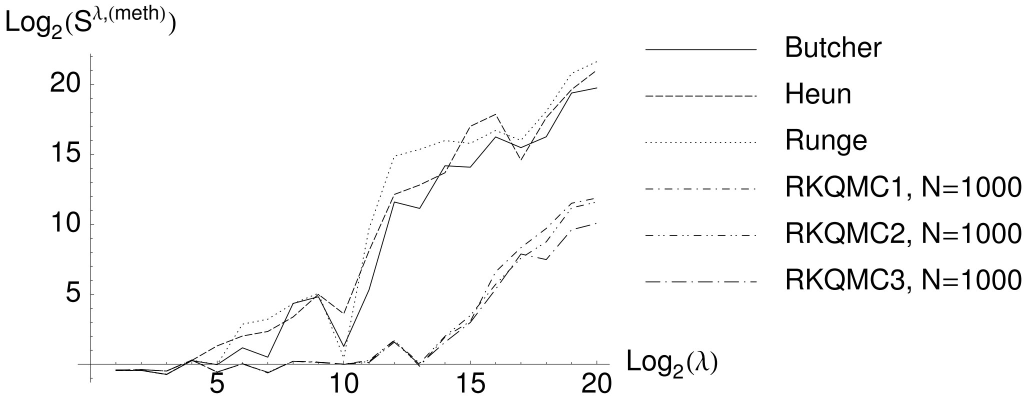

RKQMC schemes can delay the instability of the solution for heavily oscillating delay differential equations.

Time-corrected error

QMC integration is more expensive than 1 evaluation (Runge-Kutta)

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$\rightarrow$ Compare a time-corrected error

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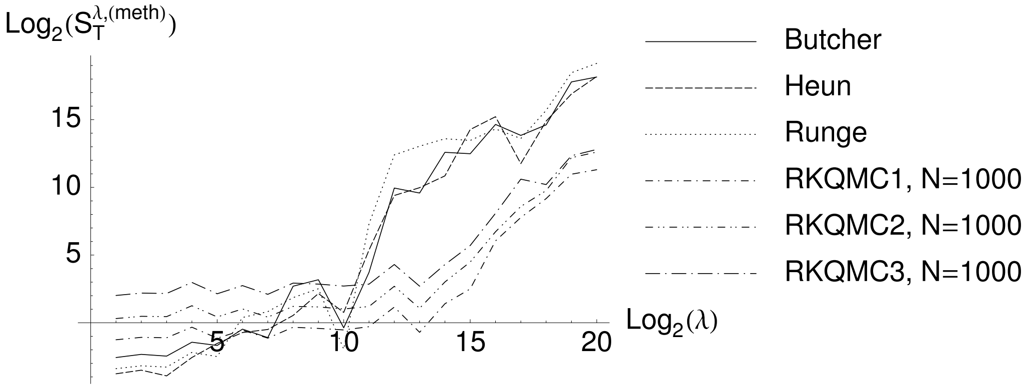

Fig: Time-corrected error of the RK and RKQMC methods

Result: RKQMC methods loose some advantage, but still better than Runge-Kutta for heavily oscillating DDE.

Non-uniform QMC integration of singular integrands

1. Convergence theorem
2. Sketch of proof
3. Hlawka-Mück construction for h-distributed low-discrepancy sequences
4. Numerical example (pricing of Asian options)

H-Diskrepancy

H-discrepancy of a sequence $\omega = (y_1, y_2, \dots)$:

$$D_{N,H}(\omega) = \sup_{J \subseteq L} \left| \frac{1}{N} A_N(J, \omega) - H(J) \right| ,$$

with $L \dots$ support of H,
$J \dots$ intervals $\left[\vec{a}, \vec{b}\right]$, $A_N \dots$ number of elements of ω lying in J.

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Koksma-Hlawka inequality (arbitrary distribution)

Let f a function of **bounded variation** (in the sense of Hardy and Krause) on L and $\omega = (y_1, y_2, \dots)$ a sequence on L. Then

$$\left| \int_L f(x) dH(x) - \frac{1}{N} \sum_{n=1}^{N} f(y_n) \right| \leq V(f) D_{N,H}(\omega).$$

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Theorem[Hartinger, K., Tichy, 2003] Let $f(x)$ be a function on $L = [a, b]$ with singularities only at the left boundary of the definition interval (i.e. $f \to \pm\infty$ only if $x^{(j)} \to a_j$ for at least one j),

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$$c_{N,j} = \min_{1 \le n \le N} y_n^{(j)} \qquad \wedge \qquad a_j < c_j \le c_{N,j}.$$

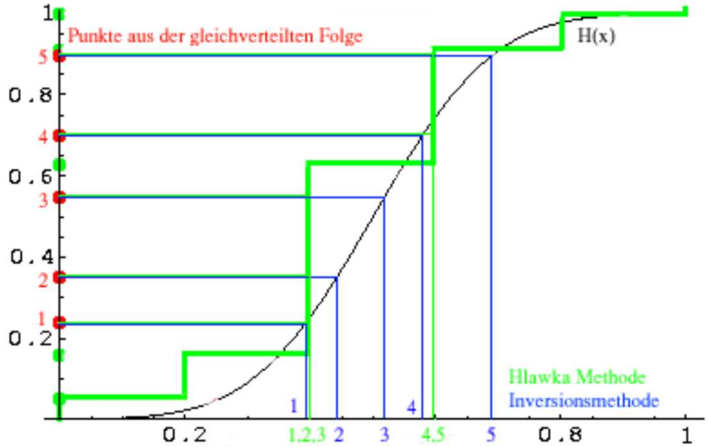
If the improper integral exists, and if

$$D_{N,H}(\omega) \cdot V_{[c,b]}(f) = o(1),$$

then the QMC estimator converges to the value of the improper integral:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(y_n) = \int_{[a,b]} f(x)\, \mathrm{d}H(x).$$

Hlawka-Mück Transformation (1 dimension)



$$\tilde{y}_k = \frac{1}{N} \sum_{r=1}^{N} \lfloor 1 + x_k - H(x_r) \rfloor = \frac{1}{N} \sum_{r=1}^{N} \chi_{[0,x_k]}(H(x_r))$$

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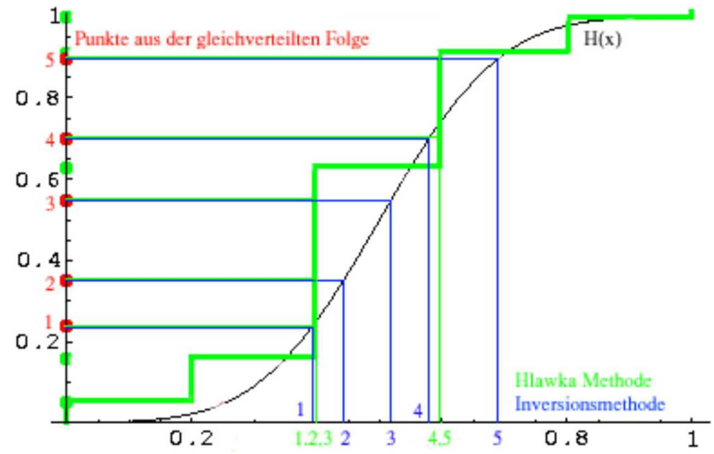

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$$D_{N,H}(\tilde{\omega}) \leq (1 + 2sM) D_N(\omega)$$

0 might appear among the $\tilde{y}_k$ $\Rightarrow$ adapt the construction as follows (needed for singular integrands): We define the sequence $\bar{\omega}$ for $i = 1, \ldots, N$ as:

$$\bar{y}_k = \begin{cases} \tilde{y}_k & \text{wenn } \tilde{y}_k \geq \frac{1}{N}, \\ \frac{1}{N} & \text{wenn } \tilde{y}_k = 0. \end{cases}$$

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The discrepancy of this sequence can be bounded as

$$D_{N,H}(\bar{\omega}) \leq (1 + 4M)^s D_N(\omega).$$

Sample problem: Valuing Asian options

arithmetic mean until expiration time

Pay-Off (discrete Asian option, call)

$$P(S_T) = \left(\frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+$$

$(S_t)_{t \geq 0}$ ... price process, K ... strike price

$S_t = e^{X_t}$ with Levy process $(X_t)_{t \geq 0}$

Increments $h_i = X_i - X_{i-1}$ with distribution H
(e.g. NIG, Variance-Gamma, Hyperbolic, ...)

NIG distribution

Use the NIG distribution for the increments $h_i \sim H^Q$.
Advantage: closed under convolution $\Rightarrow$ dimension reduction, sample only weekly instead of daily

Valuation

Using fundamental theorem (Schachermayer):

$$C_{t_0} := e^{-r(t_n - t_0)} \mathbb{E}^Q \left[\left(\frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ \right]$$

r ... constant interest rate
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Straigforward simulation (crude Monte Carlo): sample N price pathes and take the mean

Quasi-Monte Carlo schemes

Problem: QMC numbers $\overset{i.i.d.}{\sim} NIG(\alpha, \beta, \delta, \mu)$

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2 Solutions:

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   $\Rightarrow$ direct QMC calculation of the expectation value

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2 Solutions:

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2. Transformation of the integral using a suitable density (Ratio of uniforms, "Hat") $\Rightarrow$ variance reduction (if done right)

Transformation

Using a distribution $K(\vec{x}) = u$:

$$\int_{\mathbb{R}^n} P(\vec{x}) f_H^Q(\vec{x}) d\vec{x} = \int_{[0,1]^n} P(K^{-1}(u)) \frac{f_H^Q(K^{-1}(u))}{k(K^{-1}(u))} du$$

Problem: "Usual" transformation $F(x) = u$ leads to unbounded variation

Discrepancy for Hlawka-Mück

The discrepancy of $(y_k)_{1\leq k\leq N}$ can be bounded by

$$D_N\left((y_k),\rho\right) \leq (2+6sM(\rho))D_N\left((y_k)\right)$$

with $M(\rho) = \sup\rho$.

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QMC estimator

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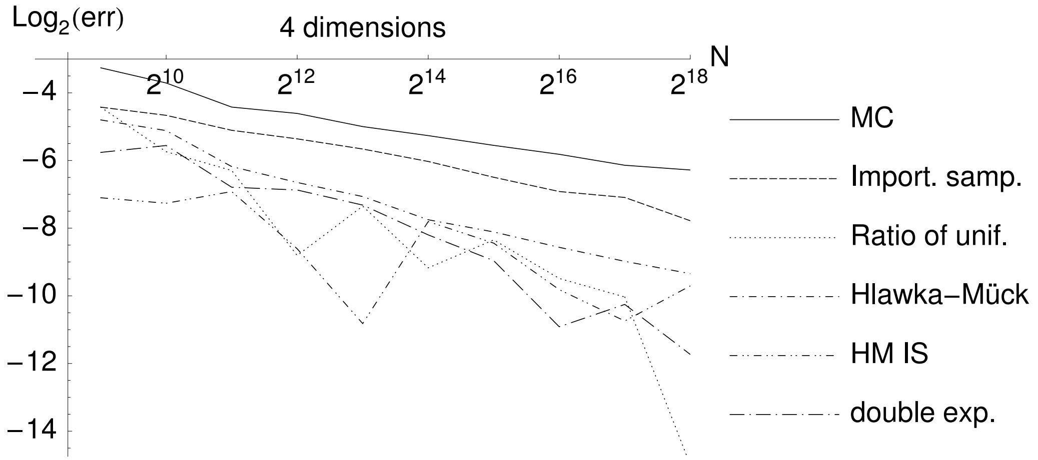
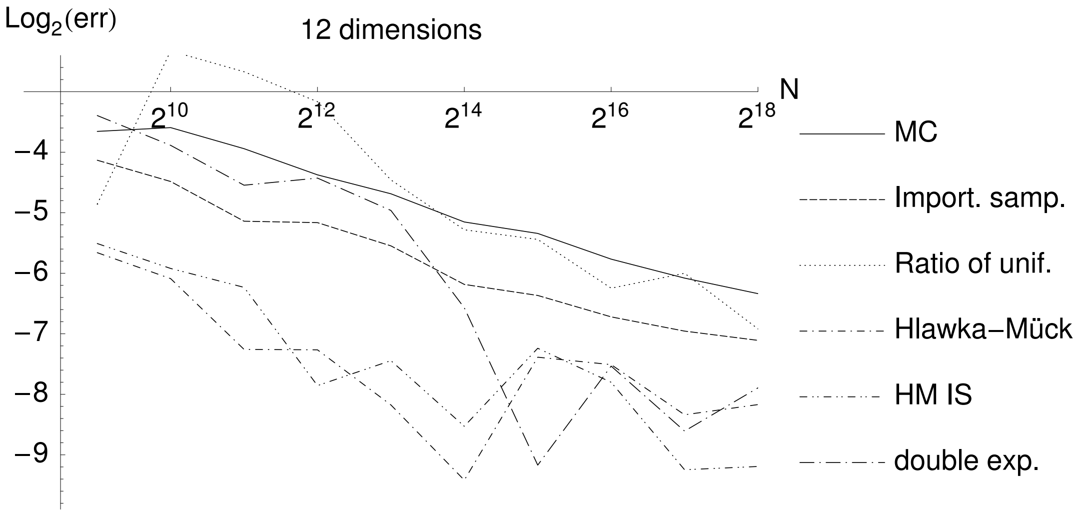
Estimator similar to crude Monte Carlo

Numerical results (4 dimensions)



ROU and Hlawka-Mück are considerably better than Monte Carlo and control variate

Dimension 12



- ROU looses performance
- Hlawka-Mück gives best results