Corner avoidance properties of various low discrepancy sequences

Reinhold Kainhofer

Financial and Actuarial Mathematics
Vienna University of Technology

Joint work with:

Volker Ziegler
Graz University of Technology

Jürgen Hartinger
Kärtner Landesversicherung

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Why do we need corner avoidance? Because of singularities there!!!

Pricing derivatives: (Fundamental theorem; Delbaen, Schachermayer (1994))

$$\widetilde{V}_t = \mathbb{E}_{\mathbb{Q}}\left[e^{-rT} C_T(\{S_t\}) \middle| \mathcal{F}_t \right]$$

with path-dependent Payoff C_T, share price process $(S_t)_{0 \le t \le T}$, interest rate t and equivalent martingale measure $\mathbb{Q}$.

Asian option: With $t \le t_1 < t_2 < \cdots < t_s \le T$, the payoff at time T of an Asian option with strike K is

$$C_T = \left(\frac{1}{s} \sum_{i=1}^{s} S_{t_i} - K \right)^+ = \left(\frac{1}{k} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} x_j} - K \right)^+$$

Increments $\mathbf{x}$ have joint distribution H (under $\mathbb{Q}$), often independent increments:

$$\mathrm{H}(\mathbf{x}) = \mathrm{H}_1(x_1)\mathrm{H}_2(x_2) \cdots \mathrm{H}_s(x_s)$$

Because of singularities there!!!

Transformation from $\mathbb{R}^d$ back to $[0,1]^d$ by componentwise inversion method ($\Rightarrow$ copula, for independent increments the independence copula results):

Fair price/value at time t

$$\widetilde{V}_t = \frac{1}{e^{rT}} \int_{\mathbb{R}^s} \left(\frac{S_0}{s} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} x_j} - K \right)^+ d\mathrm{H}(\mathbf{x})$$

$$\overset{\text{Inv.}}{=} \frac{1}{e^{rT}} \int_{[0,1]^s} \left(\frac{S_0}{s} \sum_{i=1}^{s} e^{\sum_{j=1}^{i} H_j^{-1}(1-x_j)} - K \right)^+ dx$$

Problem: $e^{H_j^{-1}(1-x_j)}$ is unbounded at $x_j = 0$!

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Extensions of Koksma-Hlawka to unbounded integrands

Despite the unbounded integrand, a convergence order can be shown for QMC:

Theorem (Owen (2006); Hartinger, K. (2005) for general distributions H)

Let $f : U^s \mapsto \mathbb{R}$ satisfy $|\partial^u f| \leq B \prod_{i=1}^s x_i^{-A_i - \chi_u(i)}$ with $A_i > 0$ for all $u \subseteq \{1, \ldots, s\}$. Let the sequence $(\mathbf{x}_n)_{1 \leq n}$ fulfill

$$\prod_{i=1}^{s} x_{n,i} \geq cN^{-r} \qquad (\text{"Origin-avoidance"}) . \tag{1}$$

Then

$$\left| \int_{U^s} f(\mathbf{x}) d\mathbf{x} - \frac{1}{N} \sum f(\mathbf{x}_n) \right| \leq C_1 D_N^*(\mathbf{x}_1, \ldots, \mathbf{x}_N) N^{r \max_i A_i} + C_2 N^{r(\max_i A_i - 1)}$$

- Similar theorem for other corners (**1** and mixed corners)
- New Problem: Need sequences with $\prod_{i=1}^s x_{n,i} \geq cN^{-r}$ with r as small as possible!
- Eqn. (1) means that the sequence avoids a hyperbolically shaped region around the origin.

Existing results about Origin Avoidance

The origin $x_0 = 0$ is left out in our considerations!

- $r \geq 1$ obvious from the construction of (t, s)-nets and Halton sequences.
- Sobol' sequences:

Theorem (Sobol' (1973), in Russian)

The product of the coordinates of the elements $\mathbf{x}_n$ of the Sobol' Π_τ sequence in dimension d for $1 \leq n < 2^\nu$ fulfills

$$x_{n,1} \cdot \ldots x_{n,d} \geq 2^{-(\nu+d+\tau)} \quad (\Rightarrow r = 1)$$

- Halton sequences:

Theorem (Owen (2005))

For $n \geq 1$, let x_n be the n'th point in a Halton sequence with distinct prime bases $p_1, \ldots, p_d$. Then

$$\prod_{j=1}^{d} x_n^j \geq \frac{1}{n} \prod_{j=1}^{d} p_j^{-1} \qquad \text{and} \qquad \prod_{j=1}^{d} \left(1 - x_n^j\right) \geq \frac{1}{n+1} \prod_{j=1}^{d} p_j^{-1}$$

Origin avoidance of generalized Niederreiter sequences

Theorem (Hartinger, K., Ziegler (2005))

Let $(\mathbf{x}_n)_{n\geq 1}$ *be a generalized Niederreiter* (t,s)*-sequence (using Tezuka's construction) and* $0 < n < b^l$ *then*

$$\prod_{i=1}^{s} x_n^{(i)} \geq b^{-l-s-t}. \qquad (\Rightarrow r = 1)$$

Sobol', Faure and Niederreiter sequences are special cases of Tezuka's construction!

Idea of the proof.

$(0,s)$**-sequences (e.g. Faure):** $(\mathbf{x}_n)_{0\leq n<b^l}$ forms a $(0,l,s)$-net. Cover the region $\prod_{i=1}^{s} x_n^{(i)} < b^{-l-s}$ with elementary intervals of volume b^{-l} that include $\mathbf{0}$ (and by definition only one element of the sequence).

(t,s)**-sequences:** $(\mathbf{x}_n)_{0\leq n<b^l}$ forms a (t,l,s)-net. There are elementary intervals $[\mathbf{0},\mathbf{a})$ of volume b^{-l} that only one (!) element of the sequence. (Proof follows along the lines of a lemma by Tezuka and Sobol's proof for the Sobol sequence)

This one element is $\mathbf{x}_0 = \mathbf{0}$, so all other elements lie outside the hyperbolic region. □

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Corner Avoidance, Motivation and Definitions

- Often integrands have singularities not only at one corner / along one boundary (e.g. "bathtube shape" $\frac{1}{x(1-x)}$)

Definition (Mixed corners and distance to corners)

Let $\mathbf{h} \in \{0,1\}^s$ be any corner of the unit cube. If $\mathbf{h} \neq \mathbf{0}$ and $\mathbf{h} \neq \mathbf{1}$, we call $\mathbf{h}$ a mixed corner. The minimum distance of the first N elements of the sequence $(x_n)_{n\in\mathbb{N}}$ to the corner $\mathbf{h}$ is

$$M_N(h_1,\dots,h_s) = \min_{1\le n\le N} \prod_{i=1}^{s} \left| h_i - x_n^{(i)} \right|$$

We use the notation:

$$J = \{i \in \{1,\dots,s\} : \; h_i{=}0\}\,, \qquad K = \{i \in \{1,\dots,s\} : \; h_i{=}1\}\,.$$

Definition

The corner avoidance coefficient $r(\mathbf{h})$ for corner $\mathbf{h} \in \{0,1\}^s$ is defined by:

$$M_N(\mathbf{h}) \geq cN^{-r(\mathbf{h})}$$

- What is $r(\mathbf{h})$ for the various low-discrepancy sequences?
- $r(\mathbf{h}) \geq 1$ clear for Halton and $(0,s)$-sequences.
- Is $r(\mathbf{h})$ independent of the corner, i.e. do the sequences tend towards all corners in a similar manner?
- For $h = \mathbf{0}$ we have $r(\mathbf{0}) = 1$ for Halton and generalized Niederreiter sequences.

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The Halton sequence avoids mixed corners less than the origin

Theorem (Owen (2006))

$$M_N(\mathbf{1}) \geq \frac{c}{N+1} \qquad \textit{and} \qquad M_N(\mathbf{h}) \geq \frac{c}{N(N+1)} \quad \textit{for } \mathbf{h} \neq \mathbf{0} \textit{ and } \mathbf{h} \neq \mathbf{1}$$

Thus, $r(\mathbf{1}) = 1$ *and* $1 \leq r(\mathbf{h}) \leq 2$.

Theorem (Hartinger, K., Ziegler (2005))

For the Halton sequence in relatively prime bases $p_1, \ldots, p_s$ *there exist subsequences* $\mathbf{y}_n = \mathbf{x}_{N(n)}$ *for which the minimum distance to any mixed corner* $\mathbf{h}$ *is bounded from above by*

$$M_{N(n)}(\mathbf{h}) \leq \frac{C}{N \log N}.$$

In particular, the corner avoidance of the Halton sequence cannot be of order $\frac{1}{N}$*, at best* $\frac{1}{N \log N} \approx \frac{1}{N^{1+\varepsilon}}$.

Proof by a simple construction of $N(n)$ using Euler's theorem.

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Corner avoidance of the Halton sequence is $r(\mathbf{h}) = 1 + \varepsilon$

Theorem (Hartinger, K., Ziegler (2005))

For the Halton sequence in relatively prime bases $p_1, \ldots, p_s$ we have

$$M_N = \min_{\mathbf{h}} M_N(\mathbf{h}) \geq \frac{c}{N^{1+\varepsilon}} .$$

Proof.

It is easy to see that:

$$\left.\begin{array}{ll} n \equiv 0 & \mod (p_i^{\alpha}) \\ n \neq 0 & \mod (p_i^{\alpha+1}) \end{array}\right\} \Rightarrow \ldots x\underbrace{0\ldots0}_{\alpha} \Rightarrow p_i^{-\alpha-1} \leq \Phi_n(p_i) = x_n^{(i)} < p_i^{\alpha},$$

$$\left.\begin{array}{ll} n \equiv -1 & \mod (p_i^{\beta}) \\ n \neq -1 & \mod (p_i^{\beta+1}) \end{array}\right\} \Rightarrow \ldots x\underbrace{1\ldots1}_{\beta} \Rightarrow p_i^{-\beta-1} \leq 1 - \Phi_{p_i}(n) = x_n^{(i)} < p_i^{\beta},$$

Proof (Continued).

Problem: Given N, "find" an integer n, with $0 < n \leq N$ and

$$n = C \prod_{j \in J} p_j^{\alpha_j} \text{ and } n + 1 = \tilde{C} \prod_{k \in K} p_k^{\beta_k}, \tag{2}$$

which minimizes

$$d = \prod_{k \in K} p_k^{-\beta_k} \prod_{j \in J} p_j^{-\alpha_j}.$$

By subtracting the equations (2) we get the diophantane equation

$$1 = \tilde{C} \prod_{k \in K} p_k^{\beta_k} - C \prod_{j \in J} p_j^{\alpha_j}$$

with unknown integers C, $\tilde{C}$, β_k, and α_j.

Example. Let $s = 2$, $p_1 = 2$, $p_2 = 3$ and consider the corner $(0, 1)$. The equation then is

$$1 = \tilde{C} 3^{\beta} - C 2^{\alpha}.$$

Schmidt's subspace theorem

Theorem (Subspace Theorem)

Let K be an algebraic number field and let $S \subset M(K) = \{$canonical absolute values of $K\}$ be a finite set of absolute values which contains all of the Archimedian ones. For each $\nu \in S$ let $L_{\nu,1}, \cdots, L_{\nu,n}$ be n linearly independent linear forms in n variables with coefficients in K. Then for given $\delta > 0$, the solutions of the inequality

$$\prod_{\nu \in S} \prod_{i=1}^{n} |L_{\nu,i}(\mathbf{x})|_{\nu}^{n_\nu} < \overline{|\mathbf{x}|}^{-\delta}$$

with $\mathbf{x} \in \mathfrak{a}_K^n$ and $\mathbf{x} \neq \mathbf{0}$, where

$$\overline{|\mathbf{x}|} = \max_{\substack{1 \leq i \leq n \\ 1 \leq j \leq \deg K}} |x_i^{(j)}|,$$

$|\cdot|_\nu$ denotes valuation corresponding to ν, n_ν is the local degree and $\mathfrak{a}_K$ is the maximal order of K, lie in finitely many proper subspaces of K^n.

Corner avoidance of Halton sequence is $r(\mathbf{h}) = 1 + \varepsilon$ (cont.)

Proof (Continued).

Use $x_1 = N + 1$ and $x_2 = N$.

Case 1: If $\frac{\widetilde{C}C}{x_2} \geq N^{-\varepsilon}$, it's straightforward to show that $d \geq (N+1)^{-1-\varepsilon}$ and our result follows.

Case 2: If $\frac{\widetilde{C}C}{x_2} < N^{-\varepsilon}$, the subspace theorem proves (by using appropriate linear forms $L_{\nu,1}(x,y)$, $L_{\nu,2}(x,y)$ and $L_{\infty,2}(x,y)$ and valuations corresponding to the primes $p_1, \ldots, p_s$ and the Archimedian valuation) that all solutions lie in finitely many proper subspaces of $\mathbb{Q}^2$, i.e.

$$x_1 a + x_2 b = 0\,.$$

Together with $x_1 - x_2 = 1$ it follows that this case happens only finitely often. Thus its effect is absorbed in the constant. □

- All-corner avoidance of the Halton sequence is almost $\mathcal{O}\left(\frac{1}{n}\right)$.
- Proof is highly non-constructive!
- Value / Bound for the constant is not known.

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Faure sequences behave differently!

- From the definition of the s-dimensional Faure sequence (Pascal matrix) in base p follows that a coordinate shift $(x_1, \ldots, x_s) \mapsto (x_2, \ldots, x_s, x_1)$ corresponds to a permutation of the elements with $p^l \leq n < p^{l+1}$.

Theorem (Corner avoidance of mixed corners; Hartinger, K., Ziegler (2005))

Let s be prime and $\mathbf{h}$ be a mixed corner (i.e. $\mathbf{h} \neq \mathbf{0}$ and $\mathbf{h} \neq \mathbf{1}$). There exists a subsequence $\mathbf{y}_n = \mathbf{x}_{N(n)}$ of the Faure sequence, such that

$$\prod_{i=1}^{s} |h_i - y_{n,i}| \leq \frac{p^3}{N(n)^2}. \tag{3}$$

Thus, we have a bound $r(\mathbf{h}) \geq 2$.

Constructive proof: Consider the subsequence $N(n) = (p-1)p^{p^n - 1}$ and possibly shift the coordinates to have $h_1 = 0$, $h_p = 1$. □

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Theorem (Corner avoidance at 1; Hartinger, K., Ziegler (2005))

Let $\mathbf{h} = \mathbf{1}$. There exists a subsequence $\mathbf{y}_n = \mathbf{x}_{N(n)}$ of the Faure sequence such that

$$\prod_{i=1}^{s}(1 - y_{n,i}) \leq \begin{cases} \frac{p^2}{N(n)^2} & \text{for } s = 2 \\ \frac{p^3}{N(n)^{3/2}} & \text{for } s > 2 \end{cases}$$

Thus, we have the bounds $r(\mathbf{1}) \geq 2$ for $s = 2$ and $r(\mathbf{1}) \geq \frac{3}{2}$ for $s > 2$.

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- These theorems show that the Faure sequence tends towards the mixed and upper corners significantly faster than the Halton sequence.

Theorem (Corner avoidance at $\mathbf{1}$; Hartinger, K., Ziegler (2005))

Let $\mathbf{h} = \mathbf{1}$. *There exists a subsequence* $\mathbf{y}_n = \mathbf{x}_{N(n)}$ *of the Faure sequence such that*

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Thank you for your attention!