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Quasi-Monte Carlo Runge Kutta methods for delay differential equations

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Section 20: Stiff Problems and Delay Differential Equations

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The problem

Delay differential equation (DDE) with one retarded argument:

$$
\begin{aligned}
y'(t) &= f\left(t, y(t), y(t-\tau(t))\right), \quad \text{for } t \geq t_0, \\
y(t) &= \varphi(t), \text{ for } t \leq t_0 ,
\end{aligned}
$$

with

$f(t,\cdot,\cdot)$ ... heavily oscillating
smooth in y and y_{ret}, but only
bounded and Borel measurable in t

$y(t)$ ... Solution,
d-dimensional real-valued function

$\tau(t)$ ... delay function, satisfying
$t_1 - \tau(t_1) \leq t_2 - \tau(t_2)$ for $t_1 \leq t_2$

$\varphi(t)$ ... initial function, piecewise continuous
on $\left(\inf_{t_0 \leq t}(t-\tau(t)), t_0\right)$

Sketch of the numerical solution

- Hermite interpolation for retarded argument $\Rightarrow$ ODE

- use RKQMC methods for ODE

Hermite interpolation

Use hermite interpolation for the retarded argument:

$$z(t) := y(t-\tau(t)) = \begin{cases} \varphi(t-\tau(t)), & \text{if } t-\tau(t) \leq t_0 \\ P_q(t-\tau(t); (y_i); (y_i')) & \text{otherwise} \end{cases}$$

DDE transforms to a ODE:

$$\begin{aligned} y'(t) = f(t, y(t), y(t-\tau(t))) \approx \\ \approx f(t, y(t), z(t)) =: g(t, y(t)) \, . \end{aligned}$$

Runge Kutta Quasi-Monte Carlo methods

G. Stengle, Ch. Lécot, I. Coulibaly, A. Koudiraty

$$y'(t) = f(t, y(t)), \qquad 0 < t < T,$$
$$y(0) = y_0$$

f smooth in y, bounded and Borel measurable in t.

f is Taylor-expanded only in y
$\rightarrow$ integral equation in t.

e.g. 2$^{\text{nd}}$ order:

$$y(t_{n+1}) = y(t_n) + \frac{1}{2h_n}\int_{t_n}^{t_{n+1}}\int_{t_n}^{t_{n+1}} \Big(f(\underline{u}, y(t_n)) + \frac{1}{\beta} f(\bar{u}, y(t_n)) + \frac{1}{\alpha} f(\bar{u}, y(t_n) + \alpha h_n f(\underline{u}, y(t_n)))\Big)\, du$$

Solved by (Quasi-)Monte Carlo integration

Quasi-Monte Carlo methods

Integrals replaced by a discrete sum over N (quasi-)random points:

$$\int_{[0,1]^s} f(x)dx = \frac{1}{N}\sum_{i=1}^{N} f(x_i)$$

MC methods: x_i random points
QMC methods: x_i low-discrepancy sequences

Low discrepancy sequences: deterministic point sequences $\{x_i\}_{1\le n\le N} \in [0,1)^s$, good uniform distribution

star-discrepancy of the point set S

$$D_N^*(S) = \sup_{a,b\in[0,1]^s}\left|\frac{A([0,1)\,,S)}{N} - \lambda_s([a,b))\right|$$

Koksma-Hlawka inequality:

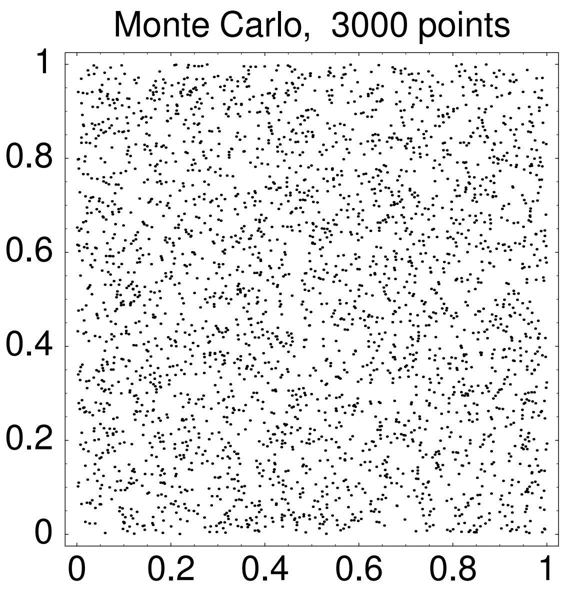
$$\left|\frac{1}{N}\sum_{n=1}^{N} f(x_n) - \int_{[0,1)^s} f(u)du\right| \le V([0,1)^s, f)D_N^*\,(x_1,\ldots,x_N)\ .$$

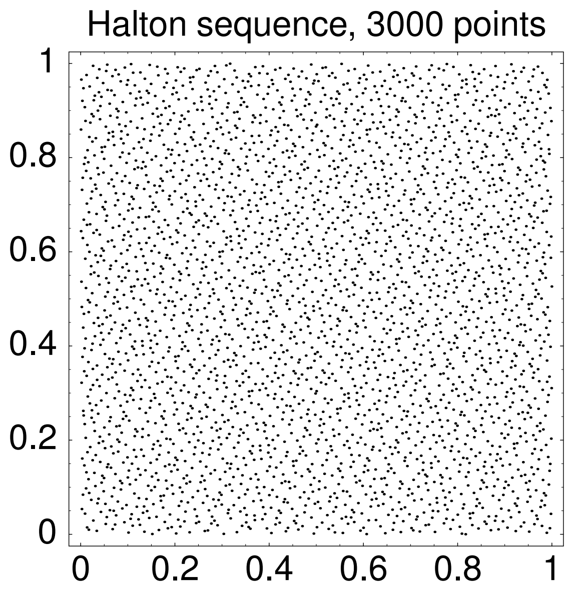
Low-Discrepancy sequences

$$D_N^*(S) \le \mathcal{O}\left(\frac{(\log N)^s}{N}\right)$$

- Halton-sequence in bases $(b_1, \dots b_s)$: inversion of digit expansion of n in base b_i at the comma

- (t, s) nets in base b (Niederreiter, Sobol, Faure): net-like structure → best possible uniform distribution on elementary intervals





Problem: correlations between elements

general form of RKQMC method

$$y_{n+1} = y_n + \frac{h_n}{s!N} \sum_{0 \le j < N} G_s \left(\bar{t}_{j,n}; y \right)$$

$G_s \left(\bar{t}_{j,n}; y \right) \ldots$ differential increment function of scheme

$$G_1 \left(u; y \right) = f \left(u, y \right)$$
$$G_2 \left(\bar{u}; y \right) = f \left(\bar{u}_1, y \right) + \frac{1}{\beta} f \left(\bar{u}_2, y \right)) + \frac{1}{\alpha} f \left(\bar{u}_2, y + \alpha h_n f \left(\bar{u}_1, y \right) \right))$$
$$G_3 \left(\bar{u}; y \right) = a_1 f(\bar{u}_1, y) + \sum_{l=1}^{L_2} a_{2,l} f \left(\bar{u}_2, y + b_{2,l} h_n f(\bar{u}_1, y) \right) +$$
$$+ \sum_{l=1}^{L_3} a_{3,l} \left(\bar{u}_3, y + b_{3,l}^{(1)} h_n f \left(\bar{u}_1, y \right) + b_{3,l}^{(2)} h_n f \left(\bar{u}_2, y_n + c_{3,l} h_n (\bar{u}_1, y_n) \right) \right)$$

Convergence proof by Koudiraty and Lécot:
If initial error $\|e_0\|$, the step size H, the discrepancy $D_N^*(X)$ and $\|f\|_E$ are small enough,

$$\|e_n\| \le e^{c_2 t_n} \|e_0\| + \frac{e^{c_2 t_n} - 1}{c_2} \left(c_1 H^3 + c_3 D_N^*(X) \right)$$

Convergence theorem, RKQMC for DDE

Choose the RKQMC method:

- G_s Lipschitz ($\mathcal{L}_2$) in 2nd and 3rd argument, bounded variation (in the sense of Hardy and Krause).

- $\exists\ c_1$, c_2, c_3 such that for a $p > 0$

$$\begin{aligned} \text{loc. trunc. error } \|\epsilon_n\| &\leq c_1(h_n)h_n^p \\ \text{RK error } \|\delta_n\| &\leq c_2(h_n)\|e_n\| \\ \text{QMC error } \|d_n\| &\leq c_3(h_n)D_N^*(S) \end{aligned}$$

- interpolation order p, fulfils a certain Lipschitz condition

Then the error $\|e_n\| = \|y_n - y(t_n)\|$ of the method is bounded from above by:

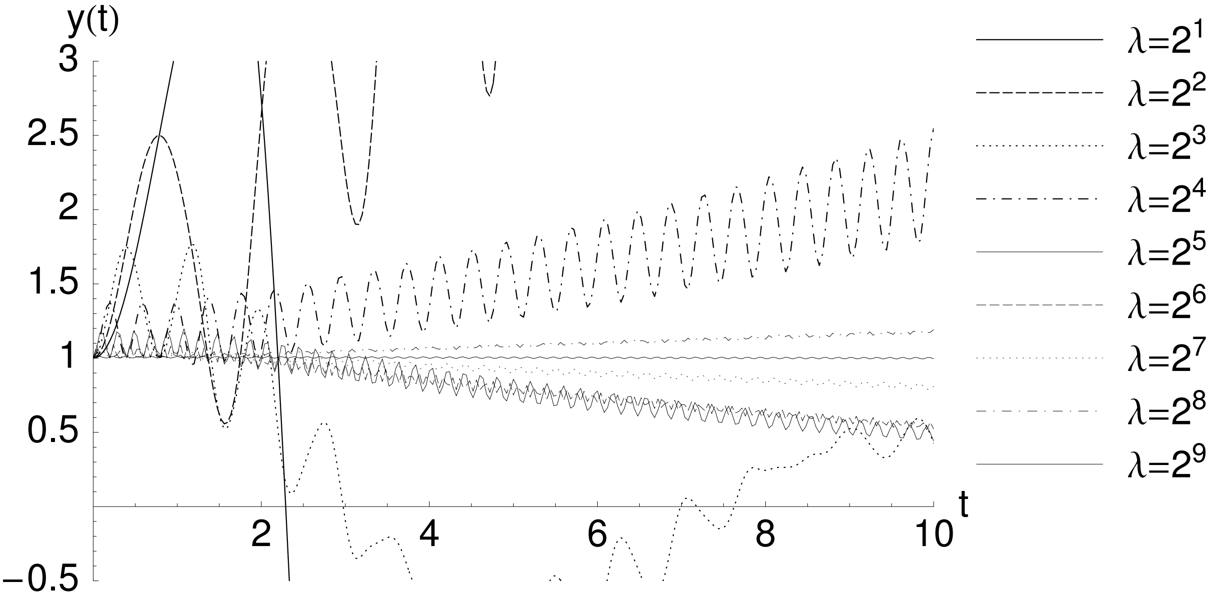
$$\|e_n\| \leq \|e_0\| e^{t_n\left(c_2+\frac{\mathcal{L}_2}{s!}\right)} + \frac{e^{t_n\left(c_2+\frac{\mathcal{L}_2}{s!}\right)} - 1}{c_2 + \frac{\mathcal{L}_2}{s!}} \cdot \left\{c_3 D_N^*(X) + \frac{\mathcal{L}_2}{s!} M H^q + c_1 H^p\right\}$$

One example (results)

$$y'(t) = 3y(t-1)sin(\lambda t), \qquad \text{for } t \geq 0$$
$$y(t) = 1, \qquad \text{for } t \leq 0$$

with $\lambda = 2^{\nu}$ and $1 \leq \nu \leq 16$.

Exact solution:



Comparison of numerical errors

slowly varying: conventional RK better
rapidly varying: RKQMC outperform higher order RK

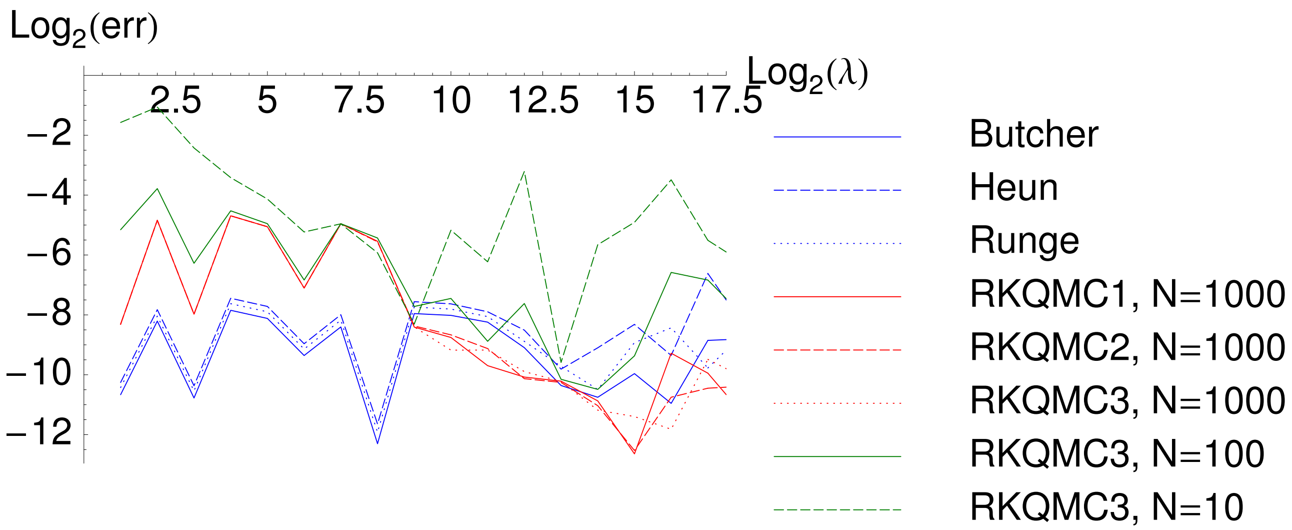


Fig: Error of the RK and RKQMC methods

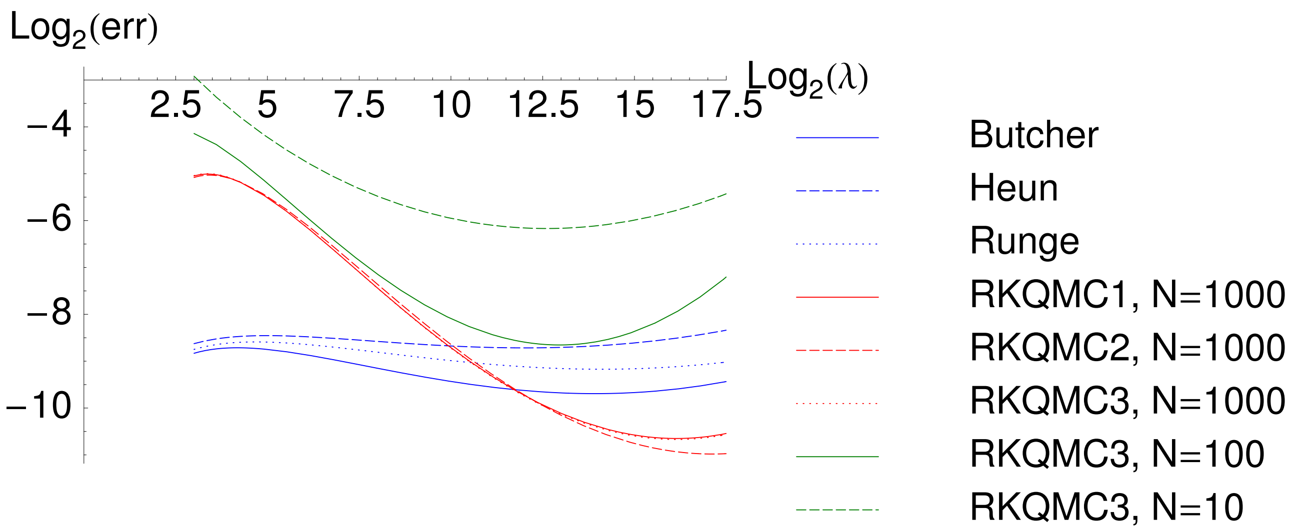


Fig: Least squares fit to the error

$h_n = 0.001$ for RK, $h_n = 0.01$ for RKQMC

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