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Hlawka-Mück techniques for option pricing
Quasi-Monte Carlo methods with NIG distribution

joint work with J. Hartinger and M. Predota

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Overview

- Sample problem: Valuing Asian options
- Crude Monte Carlo simulation
- Quasi-Monte Carlo estimators
  - Integral transformation
  - Ratio of uniforms
  - Hlawka-Mück’s method for density $f^Q$
- Numerical comparison
Sample problem: Valuing Asian options

arithmetic mean until expiration time

Pay-Off (discrete Asian option, call)

\[ P(S_T) = \left( \frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ \]

\((S_t)_{t \geq 0} \) ... price process, \( K \) ... strike price

\( S_t = e^{X_t} \) with Levy process \((X_t)_{t \geq 0}\)

Increments \( h_i = X_i - X_{i-1} \) with distribution \( H \)
(e.g. NIG, Variance-Gamma, Hyperbolic, ...)

**NIG distribution**

Use the NIG distribution for the increments $h_i \sim H^Q$. Advantage: closed under convolution $\Rightarrow$ dimension reduction, sample only weekly instead of daily

**Valuation**

Using fundamental theorem (Schachermayer):

$$C_{t_0} := e^{-r(t_n-t_0)}E^Q \left[ \left( \frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ \right]$$

$r$ ... constant interest rate  
$Q$ ... equivalent martingale measure (Esscher measure)
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Valuation

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$$C_{t_0} := e^{-r(t_n-t_0)}\mathbb{E}^Q \left[ \left( \frac{1}{n} \sum_{i=1}^{n} S_{t_i} - K \right)^+ \right]$$

$r$ ... constant interest rate
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Crude Monte Carlo simulation

Direct simulation of the process, arithmetic mean over $L$ pathes:

1. Simulate $L$ price pathes $\left((S_0^{(l)}, S_1^{(l)}, \ldots)\right)_{l \geq 1}$
   with $S_i = e^{X_i}$, $X_i = X_{i-1} + h_i$, $h_i \sim HQ$.
2. Calculate pay-off $P^{(l)}$ for each path $l$
3. Crude MC estimator: $\hat{C}_0 = e^{-r(t_n-t_0)} \frac{1}{L} \sum_{l=1}^{L} P^{(l)}$

Random numbers $h_i \sim HQ$ created using acceptance-rejection.
Quasi-Monte Carlo schemes

Problem: QMC numbers \( \overset{i.i.d.}{\sim} NIG(\alpha, \beta, \delta, \mu) \)
Quasi-Monte Carlo schemes

Problem: QMC numbers $i.i.d. \sim NIG(\alpha, \beta, \delta, \mu)$

2 Solutions:

1. Hlawka-Mück method for direct creation of $(x_n)_{1 \leq n \leq N} \overset{i.i.d.}{\sim} NIG$
   $\Rightarrow$ direct QMC calculation of the expectation value
Quasi-Monte Carlo schemes

Problem: QMC numbers \( i.i.d. \sim NIG(\alpha, \beta, \delta, \mu) \)

2 Solutions:

1. Hlawka-Mück method for direct creation of \((x_n)_{1 \leq n \leq N} \sim NIG\) \( \Rightarrow \) direct QMC calculation of the expectation value

2. Transformation of the integral using a suitable density (Ratio of uniforms, ”Hat”) \( \Rightarrow \) variance reduction (if done right)
Transformation

Using a distribution $K(\vec{x}) = u$:

$$\int_{\mathbb{R}^n} P(\vec{x}) f_H^Q(\vec{x}) d\vec{x} = \int_{[0,1]^n} P(K^{-1}(u)) \frac{f_H^Q(K^{-1}(u))}{k(K^{-1}(u))} du$$
Transformation

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$$\int_{\mathbb{R}^n} P(\vec{x}) f^Q_H(\vec{x}) d\vec{x} = \int_{[0,1]^n} P(K^{-1}(u)) \frac{f^Q_H(K^{-1}(u))}{k(K^{-1}(u))} du$$

Aim: Variance reduction for a suitably chosen distribution $K \Rightarrow k$ proportional to $|P \cdot f^Q_H|$. 
Transformation

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Problem: ”Usual” transformation $F(x) = u$ leads to integrand with unbound variation
Ratio of uniforms

"Hat" function, good choice for integral transformation
Hlawka-Mück

- Hlawka and Mück (1972): transformation of uniformly distributed sequences to low-discrepancy sequences with density $\rho$
- Hlawka (1997): simpler construction for densities $\rho = \rho_1(x_1)\rho_2(x_2) \ldots \rho_s(x_s)$
Hlawka-Mück

- Hlawka and Mück (1972): transformation of uniformly distributed sequences to low-discrepancy sequences with density $\rho$
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For density $\rho(x) = \rho_1(x_1) \cdots \rho_s(x_s)$ define

$$F_1(x^{(1)}) = \int_0^{x^{(1)}} \int_0^1 \cdots \int_0^1 \rho(u_1, \ldots, u_s) du_1 \cdots du_s$$

$$\vdots$$

$$F_s(x^{(s)}) = \int_0^1 \int_0^1 \cdots \int_0^{x^{(s)}} \rho(u_1, \ldots, u_s) du_1 \cdots du_s$$

Creation of net $(y_k)_{1 \leq k \leq N}$ with density $\rho$:

$$y^{(j)}_k = \frac{1}{N} \sum_{r=1}^N \left[ 1 + x^{(j)}_k - F_j(x^{(j)}_r) \right] , \quad j = 1, \ldots s, \quad k = 1, \ldots N$$
Discrepancy

The discrepancy of \((y_k)_{1 \leq k \leq N}\) can be bounded by

\[
D_N ((y_k), \rho) \leq (2 + 6sM(\rho)) D_N ((y_k))
\]

with \(M(\rho) = \sup \rho\).
Discrepancy

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QMC estimator

1. Creation of low-discrepancy points with density \(\frac{f^Q_k(K^{-1}(x))}{k(K^{-1}(x))}\)

   (Transformation of the integral \(\mathbb{R}^n\) to \([0, 1]^n\) using double-exponential distribution \(K(x)\))
Discrepancy

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QMC estimator

1. Creation of low-discrepancy points with density $\frac{f^Q_H(K^{-1}(x))}{k(K^{-1}(x))}$

   (Transformation of the integral $\mathbb{R}^n$ to $[0, 1]^n$ using double-exponential distribution $K(x)$)

2. Transformation $[0, 1]^n$ to $\mathbb{R}^n$ of the sequence using double-exponential distribution $K^{-1}(x)$.

Estimator similar to crude Monte Carlo
Numerical results

Dimension 4

ROU and Hlawka-Mück are considerably better than Monte Carlo and control variate
• ROU looses performance
• Hlawka-Mück gives best results